Percolation Theory Lecture Notes (2025/2026)

Griffin Reimerink

Contents

---

1 Introduction

1.1 The percolation model

Definition *Integer grid*

To form the **integer grid**, we equip the vertex set $\mathbb{Z}^d$ with the following edge set:

$$E(\mathbb{Z}^d) := \{uv : u, v \in \mathbb{Z}^d, \|u - v\| = 1\}$$

Definition *Percolation*

Let every edge in $\mathbb{Z}^d$ be **open** with probability p and **closed** otherwise. (i.i.d.)
We speak of a **percolation** if there exists an infinite path of open edges.
The percolation model can also be viewed as a Bernoulli distributed infinite random vector.

Definition *Cluster*

A **connected component** or **cluster** in a graph G is a set of vertices $C \subseteq V(G)$ such that

- there is a path between u, v for all $u, v \in C$
- every vertex not in C does not have a path to any vertex in C

Notation

$$\{a \leftrightsquigarrow b\} := \{\text{there is an open path between } a \text{ and } b\}$$

$$\{a \leftrightsquigarrow \infty\} := \{\text{there is an infinite open path starting in } a\}$$

$$\{A \leftrightsquigarrow B\} := \bigcup_{a \in A, b \in B} \{a \leftrightsquigarrow b\}$$

$$\{a \overset{C}{\leftrightsquigarrow} b\} := \{\text{there is an open path between } a \text{ and } b \text{ that is completely contained in } C\}$$

Definition *Box*

$$\Lambda_n = \{z \in \mathbb{Z}^d : \max_{i=1,\dots,d} |z_i| \le n\} \qquad \partial\Lambda_n = \{z \in \mathbb{Z}^d : \max_{i=1,\dots,d} |z_i| = n\}$$

Definition *Percolation function*

$$\theta(p) := \mathbb{P}_p(\underline{0} \leftrightsquigarrow \infty)$$

Lemma

$\theta(p) > 0$ if and only if $\mathbb{P}_p(\text{percolation}) > 0$.

Definition *Critical probability*

$$p_c := \inf\{p \in [0, 1] : \theta(p) > 0\}$$

1.2 Up-sets

Definition *Up-set*

An event is called an **up-set** or an **increasing event** if the following holds:
For all $a = (a_e)_{e \in \mathbb{Z}^d} \in A$ and every $F \in E(\mathbb{Z}^d)$, the vector $b = (b_e)_{e \in \mathbb{Z}^d}$ defined by

$$b_e = \begin{cases} a_e & \text{if } e \neq f \\ 1 & \text{if } e = f \end{cases}$$

is an element of A.
An event A is called a **down-set** or **decreasing event** if A^c is an up-set.

Standard increasing coupling

We can define an alternate percolation model as follows:
For each $e \in E(\mathbb{Z}^d)$, we let the random variable U_e be uniform on $[0, 1]$ and independent of all other U_f.
For $p \in [0, 1]$ let G_p be the subgraph of $\mathbb{Z}^d$ we get by keeping every e with $U_e \leq p$. Then

- G_p behaves exactly like the percolation model with parameter p.
- If $p < q$, then by construction G_p is a subgraph of G_q.

Lemma

If the event E is an up-set then $p \mapsto \mathbb{P}_p(E)$ is non-decreasing.

Corollary

The functions $p \mapsto \theta(p)$ and $p \mapsto \mathbb{P}_p(\text{percolation})$ are non-decreasing.

Theorem

The function $p \mapsto \theta(p)$ is strictly increasing on $(p_c, 1]$.

1.3 Probability of percolation is either 0 or 1

Lemma

For every $\varepsilon > 0$, there exists an event F, depending only on the status of the edges inside a finite box Λ_n, such that $\mathbb{P}(\{\text{percolation}\}\Delta F) < \varepsilon$.

Lemma

The percolation model is invariant under translations.

Theorem

For every dimension d, and every $0 \leq p \leq 1$, we have $\mathbb{P}_p(\text{percolation}) \in \{0, 1\}$.

1.4 Bounds on the percolation function

Theorem

In every dimension $d \geq 2$, the function $p \mapsto \theta(p)$ is continuous from the right for all $p \in [0, 1]$:

$$\lim_{q \searrow p} \theta(q) = \theta(p)$$

Theorem *(Broadbent, Hammersley)*

In dimension $d = 2$ we have $\frac{1}{3} \leq p_c \leq \frac{2}{3}$

Corollary

$p_c \leq \frac{2}{3}$ for all dimensions $d \geq 2$.

2 Bond percolation on $\mathbb{Z}^2$

2.1 The dual percolation model

Definition *Planar dual*

The **planar dual** G^* of a plane graph G is obtained by placing a vertex in the interior of each face, and connecting two vertices of G^* by an edge if and only if the corresponding faces of G meet along an edge.

Definition *Dual percolation model*

We can draw $\mathbb{Z}$ and $(\mathbb{Z}^2)^*$ in the same picture by putting vertices of $(\mathbb{Z}^2)^*$ on the faces of $\mathbb{Z}^2$,
and define a coupling of percolation on $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$ by setting e^* to be open if and only if e is closed.

2.2 Crossings

Definition *Horizontal and vertical crossing*

Let $a < b$ and $c < d$ be integers, and define the rectangle $R := \{a, \dots, b\} \times \{c, \dots, d\}$
We define a **horizontal crossing** and **vertical crossing** as follows:

$$H(R) := \Big\{ \{a\} \times \{c, \dots, d\} \overset{R}{\leftrightsquigarrow} \{b\} \times \{c, \dots, d\} \Big\}$$

$$V(R) := \Big\{ \{a, \dots, b\} \times \{c\} \overset{R}{\leftrightsquigarrow} \{a, \dots, b\} \times \{d\} \Big\}$$

Definition *Dual rectangle*

Let R be a $(n+1) \times n$ rectangle, and remove the edges on the left and right boundary.
Each of the remaining edges of R intersect precisely one edge of the dual. If we take all these dual edges together with the dual vertices they are incident with, then we get a copy R^* of R in the dual, but rotated by 90 degrees.

Lemma

For any rectangle $R \subset \mathbb{Z}^2$, precisely one of $H(R)$ and $V(R^*)$ holds.

Proposition

Let R be a $(n+1) \times n$. Then, independent of n, $\mathbb{P}_{1/2}(H(R)) = 1/2$.

2.3 Russo-Seymour-Welsh theorem

Theorem *Harris' inequality*

If $A, B \subset \{0,1\}^n$ are up-sets, then

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Theorem *(Russo-Seymour-Welsh)*

For every $\alpha > 0$, there exists a constant $c(\alpha) > 0$ such that

$$\mathbb{P}_{1/2}(H(R)) \geq c(\alpha)$$

for any $\lceil \alpha n \rceil \times n$ rectangle R, for all $n \in \mathbb{N}$.

Theorem *Harris' theorem*

In dimension $d = 2$ we have $\theta(1/2) = 0$.

Corollary

In dimension $d = 2$ we have $p_c \geq 1/2$.

2.4 1-independent percolation

Theorem

In dimension $d = 2$, we have for all $p \in [0, 1]$,

$$\mathbb{P}_p(\text{there exist more than one distinct infinite clusters}) = 0$$

Definition *1-independent percolation model*

A percolation model is **1-independent** if the events $\{e_1 \text{ open}\}, \dots, \{e_k \text{ open}\}$ are independent whenever the edges $e_1, \dots, e_k \in E(\mathbb{Z}^d)$ do not share endpoints (for all $k \geq 1$).

Lemma

There exists a $p_1 < 1$ such that in any 1-independent percolation model on $\mathbb{Z}^d$ with $P(e \text{ open}) \geq p_1$ for all $e \in E(\mathbb{Z}^d)$, we have

$$\mathbb{P}_p(\text{there exists an infinite cluster}) > 0$$

2.5 Kesten's theorem

Lemma

There exists a $p_0 < 1$ such that the following holds:

$$\exists\, p, n \quad \mathbb{P}_p(H(\{1, \dots, 3n\} \times \{1, \dots, n\}) \geq p_0 \implies \mathbb{P}_p(\text{percolation}) = 1$$

Lemma

For all $p > 1/2$, we have

$$\lim_{n \to \infty} \mathbb{P}_p(H(\{1, \dots, 3n\} \times \{1, \dots, n\})) = 1$$

Theorem *Kesten's theorem*

In dimension $d = 1/2$, we have $p_c = 1/2$.

3 Boolean functions

3.1 Influences

Definition *Pivotal element*

Let $A \subset \{0, 1\}^n$ be an event. We say that X_i is **pivotal** if one of

$$(X_1, \dots, X_{i-1}, 0, X_i, \dots, X_n),\ (X_1, \dots, X_{i-1}, 1, X_i, \dots, X_n)$$

is in A and the other is not.

Definition *Influence*

Let $A \subset \{0, 1\}^n$ be an event. We define the i-th **influence** as:

$$\mathrm{Inf}_i(A) := \mathbb{P}_p(i \text{ is pivotal for } A)$$

Theorem *Margulis-Russo formula*

Let $A \subset \{0, 1\}^n$ be an arbitrary up-set. Then

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \sum_{i=1}^{n} \mathrm{Inf}_i(A)$$

Theorem *Talagrand's inequality*

There exists a universal constant $c > 0$ such that for all $n \in \mathbb{N}, 0 \leq p \leq 1$ and all events $A \subseteq \{0, 1\}^n$,

$$\sum_{i=1}^{n} \mathrm{Inf}_i(A) \geq c \cdot \mathbb{P}(A) \cdot (1 - \mathbb{P}(A)) \cdot \ln\left(\frac{1}{\max\limits_{i=1,\dots,n} \mathrm{Inf}_i(A)}\right)$$

3.2 Fourier-Walsh decomposition

Definition *Boolean function*

An event $A \subseteq \{0,1\}$ corresponds with a **boolean function** $f : \{\pm 1\}^n \to \{\pm 1\}$ given by

$$f(x_1, \ldots, x_n) := \begin{cases} 1 & \text{if } \left(\frac{x_1+1}{2}, \ldots, \frac{x_n+1}{2}\right) \in A \\ -1 & \text{otherwise} \end{cases}$$

Definition

Define the following normed vector space:

$$V := \{f : \{\pm 1\}^n \to \mathbb{R}\} \qquad \langle f, g \rangle = \mathbb{E}[f(X)g(X)]$$

where $X = (X_1, \ldots, X_n)$ with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$ for all i.
Also define for $S \subseteq \{1, \ldots, n\}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$:

$$x^S := \prod_{i \in S} x_i$$

Lemma

The functions $x^s : S \subseteq \{1, \ldots, n\}$ form an orthonormal basis for V.

Definition *Fourier-Walsh decomposition*

Every $f \in V$ is a linear combination of the x^S:

$$f = \sum_{S \subseteq \{1, \ldots, n\}} \hat{f}(S) \cdot x^S$$

The coefficients $\hat{f}(S)$ are called **Fourier-Walsh coefficients**.

Lemma *Plancharel's identity*

$\langle f, g \rangle = \sum_S \hat{f}(S) \cdot \hat{g}(S)$

Corollary *Parseval's identity*

$\langle f, f \rangle = \sum_S \hat{f}(S)^2$, and if f is ± 1 valued, then $\langle f, f \rangle = 1$.

Lemma

We have $\mathbb{E}(f) = \hat{f}(\varnothing)$ and $\mathrm{Var}(f) = \sum_{S \neq \varnothing} \hat{f}(S)^2$.

3.3 Differential and expectation operator

Definition *Differential and expectation operator*

For $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $1 \leq i \leq n$, we write

$$x^{i \to y} := (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$$

For $f \in V$, we define the i-th **differential operator** and **expectation operator** as follows:

$$(D_i f)(x) := \frac{1}{2}\left(f(x^{i \to 1}) - f(x^{i \to -1})\right) \qquad (D_i f)(x) := \frac{1}{2}\left(f(x^{i \to 1}) + f(x^{i \to -1})\right)$$

Lemma

The differential and expectation operator are linear.

Lemma

For all $f \in V$, the functions $D_i f$ and $E_i f$ have Fourier-Walsh decomposition

$$D_i f = \sum_{S \ni i} \hat{f}(S) x^{S \setminus \{i\}} \qquad E_i f = \sum_{S \not\ni i} \hat{f}(S) x^S$$

Lemma

If f is ± 1 valued then $\mathrm{Inf}_i(f) = \mathbb{E}[D_i f] = \mathbb{E}(D_i f)^2$

Corollary

If f is ± 1 valued then

$$\sum_i \mathrm{Inf}_i(f) = \sum_S |S| \hat{f}(S)^2$$

where $\hat{f}(S)$ are the Fourier-Walsh coefficients of $D_i f$.

3.4 Noise stability

Definition *ρ-correlation*

For $x \in \{\pm 1\}^n$ and $\rho \in [-1, 1]$, we say $Y = (Y_1, \ldots, Y_n)$ is ρ-**correlated** to x, denoted $Y \sim N_\rho(x)$, if the Y_i are chosen independently such that

$$Y_i = \begin{cases} x_i & \text{with probability } 1/2 + \rho/2 \\ -x_i & \text{with probability} 1/2 - \rho/2 \end{cases}$$

In particular, if $Y \sim N_p(x)$, then $\mathbb{E}(Y_i) = \rho x_i$.

Definition *Noise operator and stability*

For $\rho \in [-1, 1]$, the ρ **noise operator** T_p assigns to $f \in V$ the function $T_p f$ given by

$$(T_p f)(x) = \mathbb{E}_{Y \sim N_\rho(x)} f(Y)$$

The **noise stability** is defined as

$$\mathrm{Stab}_\rho(f) := \langle f, T_p f \rangle = \mathbb{E}_X \left[\mathbb{E}_{Y \sim N_\rho(x)} f(X) f(Y) \right]$$

Lemma

If f is ± 1 valued then

$$\mathrm{Stab}_\rho f = \mathbb{P}(f(X) = f(Y)) - \mathbb{P}(f(X) \neq f(Y)) = 1 - 2\mathbb{P}(f(X) \neq f(Y))$$

where $X \sim \mathrm{Unif}(\{\pm 1\}^n)$ and $Y \sim N_\rho(X)$

Lemma

The Fourier-Walsh decomposition of $T_\rho f$ is

$$T_\rho f = \sum_S \rho^{|S|} \hat{f}(S) x^S$$

Corollary

$$\mathrm{Stab}_\rho f = \sum_S \rho^{|S|} \hat{f}(S)^2$$

Corollary

$$\sum_i \mathrm{Stab}_\rho D_i f = \sum_S \rho^{|S|-1} |S| \hat{f}(S)^2$$

3.5 Hypercontractivity

Definition *p-norm*

For $p > 0$ and $f \in V$, define the **p-norm** of f by

$$\|f\|_p := \sqrt[p]{\mathbb{E}|f|^p}$$

Proposition *Jensen's inequality*

$$\|f\|_4 \geq \|f\|_2$$

Definition *Contraction*

An operator T is called a **contraction** if $\|Tf\| \leq \|f\|$ for all f in some fixed norm $\|.\|$.

Theorem *Cauchy-Schwartz inequality*

$$\langle g, h\rangle \leq \|g\|_2 \cdot \|h\|_2$$

Theorem *(2,4)-hypercontractivity*

For all $f \in V$, we have

$$\|T_{1/\sqrt{3}} f\|_4 \leq \|f\|_2$$

Theorem *Hölder's inequality*

If $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for all f and g,

$$\langle f, g\rangle \geq \|f\|_p \cdot \|g\|_q$$

Theorem *(4/3,2)-hypercontractivity*

For all $f \in V$, we have

$$\|T_{1/\sqrt{3}} f\|_2 \leq \|f\|_{4/3}$$

Corollary

For any $f : \{\pm 1\}^n \to \{-1, 0, 1\}$ we have $\mathrm{Stab}_{1/3}(f) \leq (\mathbb{E}|f|)^{3/2}$.

Corollary

For any $f : \{\pm 1\}^n \to \{-1, 1\}$ we have $\mathrm{Stab}_{1/3}(D_i f) \leq \mathrm{Inf}_i(f)^{3/2}$.

4 Percolation on $\mathbb{Z}^d$ and other models

4.1 Infinite clusters

Notation

In this section, N denotes the number of distinct infinite open clusters in $\mathbb{Z}^d$ for any $d \geq 1$ and $p \in [0, 1]$.

Lemma

$\mathbb{P}(N \in \{0, 1, \infty\}) = 1$

Definition *Trifurcation point*

We say that $z \in \mathbb{Z}^d$ is a **trifurcation point** if $z \leftrightsquigarrow \infty$ and z has degree precisely three,
and if we remove z then the cluster of z splits into precisely three infinite clusters.

Lemma

Suppose that $\mathbb{P}(N = \infty) = 1$. Then $\mathbb{P}(\underline{0}$ is a trifurcation point$) > 0$.

Theorem *(Aizenman-Kesten-Newmann)*

For bond percolation on $\mathbb{Z}^d$ with $d \geq 1$ and any $p \in [0, 1]$ we have

$$\mathbb{P}_p(\text{there exist} \geq 2 \text{ distinct infinite open clusters}) = 0$$

Theorem *(Van den Berg)*

In all dimensions d, the function $p \mapsto \theta(p)$ is continuous on $[0, 1] \setminus \{p_c\}$.

4.2 The BK-inequality

Definition *Box operator*

Let $A, B \subseteq \{0, 1\}^n$ be up-sets. We define the event $A \square B$ (A and B occur "for disjoint reasons") as:

$$A \square B := \{(x_1, \ldots, x_n) \in \{0, 1\}^n : \exists \text{ disjoint } I, J \subseteq \{1, \ldots, n\} \text{ such that } 1_{I \cup J} \leq x \text{ and } 1_I \in A, 1_J \in B\}$$

Note: $A \square A$ means that A occurs twice.

Theorem *BK-inequality*

For all $p \in [0, 1]$, all $n \in \mathbb{N}$ and all pairs of up-sets $A, B \subseteq \{0, 1\}^n$, we have

$$\mathbb{P}(A \square B) \leq \mathbb{P}(A) \cdot \mathbb{P}(B)$$

4.3 Exponential decay

Definition *Edge boundary*

The **edge boundary** of a set $S \subseteq \mathbb{Z}^d$ is

$$\partial_E S := \{(u, v) : e = uv \in E(\mathbb{Z}^d) \text{ with } u \in S, v \notin S\}$$

Notation

$$\theta_n(p) := \mathbb{P}(\underline{0} \leftrightsquigarrow \partial\Lambda_n) \qquad \varphi(S, p) := p \cdot \sum_{(u,v) \in \partial_E S} \mathbb{P}_p\left(\underline{0} \overset{S}{\leftrightsquigarrow} u\right) \qquad T := \{z \in \Lambda_n : z \not\leftrightsquigarrow \partial\Lambda_n\}$$

Lemma

If there exists a finite $S \subseteq \mathbb{Z}^d$ with $\underline{0} \in S$ and $\phi(S, p) < 1$ then there exists a $c = c(p, d) > 0$ such that

$$\theta_n(p) \leq e^{-cn} \quad \text{for all } n$$

Lemma

For all $0 < p < 1$ we have

$$\theta_n'(p) = \frac{1}{p(1-p)} \cdot \mathbb{E}_p \varphi(T, p)$$

Theorem *(Aizenman-Barsky, Menshikov)*

In every dimension $d \geq 2$, for every $p < p_c(\mathbb{Z}^d)$ there exists a constant $c = c(p, d)$ such that

$$\mathbb{P}_p(\underline{0} \leftrightsquigarrow \partial\Lambda_n) < e^{-cn} \quad \text{for all } n$$

Corollary *(Aizenman-Barsky, Menshikov)*

In every dimension $d \geq 2$, there exists a constant $c = c(d)$ such that

$$\theta(p) \geq c(p - p_c) \quad \text{for all } p \geq p_c$$

Theorem *(Russo)*

In dimension $d = 2$, the function $p \mapsto \theta(p)$ is differentiable on $(0,1) \setminus \{p_c\}$.

Definition *Radius and volume of a cluster*

$$R := \sup\{n : \underline{0} \leftrightsquigarrow \partial\Lambda_n\} \qquad C := \left|\{z \in \mathbb{Z}^d : \underline{0} \leftrightsquigarrow z\}\right|$$

Definition *Lattice animal*

A **lattice animal** is a connected subset of $\mathbb{Z}^d$ that contains the origin.

Theorem

For every $d \geq 2$ and every $p < p_c$ there exists a $c = c(p)$ such that

$$\mathbb{P}(C \geq n) \leq e^{-cn} \quad \text{for all } n > 1$$

Theorem

Consider bond percolation in dimension $d \geq 2$.

1. If $p < p_c$ then $\lim\limits_{n\to\infty} \mathbb{P}(H(\Lambda_n)) = 0$
2. If $p > p_c$ then $\lim\limits_{n\to\infty} \mathbb{P}(H(\Lambda_n)) = 1$

4.4 Triangle and honeycomb lattices

Definition *Triangular lattice*

The vertices of the **triangular lattice** T are all integer linear combinations of $v_1 = (1,0)$ and $v_2 = (1/2, \sqrt{3}/2)$, and two vertices are connected iff their distance is exactly 1. The dual of T is the **hexagonal lattice** H.

Definition *Rhombus*

In T, the $(2n+1) \times (2n+1)$ **rhombus** is defined by

$$R_n := \{iv_1 + jv_2 : i, j \in \{-n, \dots, n\}\}$$

We define events $H(R_n)$ and $V(R_n)$ on T in the same way as in $\mathbb{Z}^2$. On H we define $H(R_n)$ to be the event that a horizontal crossing exists in the subgraph of H spanned by all edges of H that intersect some edge of R_n.

Proposition

$p_c(T) + p_c(H) = 1$

Theorem *(Wierman)*

$$p_c(T) = 2\sin(\pi/18) \qquad p_c(H) = 1 - 2\sin(\pi/18)$$

4.5 Strict inequalities

Theorem *(Van den Berg, Frieze)*

$p_c(\mathbb{Z}^{d+1}) < p_c(\mathbb{Z}^d)$ for all $d \geq 1$.

Definition *Periodic adding and removing of edges*

We say the graph G is obtained from $\mathbb{Z}^d$ by adding the edge $e = uv \notin E(\mathbb{Z}^d)$ with period $\pi \in \mathbb{N}$ if $V(G) = \mathbb{Z}^d$ and

$$E(G) = E(\mathbb{Z}^d) \cup \{(u_1 + k_1\pi, \dots, u_d + k_d\pi)(v_1 + k_1\pi, \dots, v_d + k_d\pi) : k_1, \dots, k_d \in \mathbb{Z}\}$$

The graph G is obtained from $\mathbb{Z}^d$ by removing the edge $e = uv \notin E(\mathbb{Z}^d)$ with period $\pi \in \mathbb{N}$ if $V(G) = \mathbb{Z}^d$ and

$$E(G) = E(\mathbb{Z}^d) \setminus \{(u_1 + k_1\pi, \dots, u_d + k_d\pi)(v_1 + k_1\pi, \dots, v_d + k_d\pi) : k_1, \dots, k_d \in \mathbb{Z}\}$$

Theorem *(Menshikov)*

For all $d \geq 2$ it holds that

1. If G is obtained from $\mathbb{Z}^d$ by periodically adding an edge then $p_c(G) < p_c(\mathbb{Z}^d)$.
2. If G is obtained from $\mathbb{Z}^d$ by periodically removing an edge then $p_c(G) > p_c(\mathbb{Z}^d)$.

4.6 Site percolation

Definition *Site percolation*

In **site percolation**, all the nodes are either open or closed (independently, with probability p), and the edges are always open. We now say (site) percolation occurs if there is an infinite path all of whose nodes are open.
We define the critical probability for site percolation by:

$$p_c(G, \text{site}) = \inf\{p \in [0,1] : \mathbb{P}_p(\text{site percolation}) > 0\}$$

Definition *Line graph*

For a graph G, its **line graph** has vertices $V(L(G)) = E(G)$ and the following edge set:

$$E(L(G)) := \left\{ef \in \binom{E(G)}{2} : |e \cap f| = 1\right\}$$

That is, the edge set of $L(G)$ are all pairs of edges of G that share an endpoint.

Theorem *(Hammersley, Oxley-Welsh)*

For every countable graph G, we have

$$p_c(G) \leq p_g(G, \text{site})$$

Proposition

Let G be a countable graph and $v \in V(G)$.
Let $\mathscr{C}_{\text{bond}}$ and $\mathscr{C}_{\text{site}}$ denote the cluster containing v for site and bond percolation respectively.
For every p there exists a coupling of site and bond percolation on G, both with parameter p, such that

$$\mathscr{C}_{\text{site}}(v) \subseteq \mathscr{C}_{\text{bond}}(v)$$

Theorem *(Grimmett-Stacy)*

$p_c(\mathbb{Z}^d, \text{site}) > p_c(\mathbb{Z}^d)$ for all $d \geq 2$.

4.7 Asymptotics of p_c

Theorem *(Gordon, Hara-Slade, Kesten)*

$$p_c(\mathbb{Z}^d) = (1 + o_d(1)) \cdot \frac{1}{2d}$$

where $o_d(1)$ is an error term that tends to zero as $d \to \infty$.

Definition *Galton-Watson process*

Initially, in generation $t = 0$, there is a single individual. This individual has a random number $X \in \mathbb{N}_0$ of children. Each of these children now has a random number of children themselves (distributed like X), and so on. We keep going until the process has died out (zero individuals in some generation) or, if that does not happen, then we continue indefinitely. In the first case we say the process becomes **extinct** in the second case we say it **survives**.

Theorem

Provided $0 < \mathbb{P}(X = 0) < 1$, we have $\mathbb{P}(\text{extinction}) = 1$ if and only if $\mathbb{E}(X) \geq 1$.

Definition *Infinite k-regular tree*

An **infinite k-regular tree** T_k is a countably infinite tree in which every vertex has degree exactly k.

Lemma

For bond percolation on T_k, $p_c(T_k) = \frac{1}{k-1}$.

Definition *Stochastic domination*

The random variable X stochastically dominates the random variable Y (notation $X \geq_{\text{st}} Y$) if

$$\mathbb{P}(X \geq x) \geq \mathbb{P}(Y \geq x) \quad \text{for all } x$$

Lemma

Let X, Y be integer valued random variables.
Then $X \geq_{\text{st}}$ if and only if there is a coupling (X', Y') for X and Y such that $\mathbb{P}(X' \geq Y') = 1$

Lemma

Let C be the number of nodes in the cluster of the origin in $\mathbb{Z}^d$.
For every $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ and $d_0 = d_0(\varepsilon)$ such that for all $d \geq d_0$ and $p \geq (1+\varepsilon)/2d$ we have

$$\mathbb{P}(C \geq \sqrt{d}) \geq c$$

Theorem *(Grimmett-Marstrand)*

For every $d \geq 3$ we have

$$\lim_{n \to \infty} p_c(\mathbb{Z}^2 \times \{1, \ldots, n\}^{d-2}) = p_d(\mathbb{Z}^d)$$

Index